The Casimir effect for thin plasma sheets and the role of the surface plasmons

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Abstract
We consider the Casimir force between two dielectric bodies described by the plasma model and between two infinitely thin plasma sheets. In both cases in addition to the photon modes, surface plasmons are present in the spectrum of the electromagnetic field. We investigate the contribution of both types of modes to the Casimir force and confirm resp. find in both models large compensations between the plasmon modes themselves and between them and the photon modes especially at large distances. Our conclusion is that the separation of the vacuum energy into plasmon and photon contributions must be handled with care except for the case of small separations.

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1. Introduction

The Casimir force between macroscopic bodies is at present one of the most interesting macroscopic manifestations of quantum effects. Applications range from attempts to obtain new constraints on hypothetical fifth forces in high precision experiments to impact on micromachinery and nanotechnology. The influence of the real structure of the interacting bodies is of increasing importance. For a better understanding of the Casimir forces, efforts are undertaken to investigate the contributions from separate parts of the spectrum of the electromagnetic field. For instance, it is assumed that the surface plasmons (waves propagating along the surface[1]) dominate the Casimir force at small separations while at large separations the photon modes, i.e., the waves propagating perpendicular to the surfaces dominate.

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[1] The notion of surface plasmons is sometimes used in a different definition. So in [1] it is generalized to include the photon modes too.

In the present paper, we investigate the Casimir effect for thin plasma sheets and for dielectric bodies described by the plasma model and can confirm the above assumption only partly. We find large cancellations between the plasmon modes and between the plasmon and the photon modes at all distances. For the first model this was observed already in [2].

The considered models are

(i) *Two dielectric bodies described by the plasma model.* Actually we assume two half spaces filled with a dielectric medium and separated by a plane gap of width L. The dielectric function is described by the plasma model

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}, \tag{1}$$

where ω_p is the plasma frequency.

(ii) *Thin plasma sheets.* Barton considered in [3, 4] a model of an infinitely thin shell filled with a charged fluid in order to describe a single layer hexagonal structure of carbon atoms aimed to model C_{60} or carbon nanotubes. The interaction of such plasma sheet with the electromagnetic field results in a set of matching conditions for the latter. We consider two parallel sheets at separation L.

In fact, these models result in very similar formulae. Both can be described by matching conditions and the Casimir force is given by the Lifshitz formula or a simple modification of it. Both models have surface plasmons in their TM polarization. The first model is well known and nearly all quantities considered below can be found in the literature, see e.g. [2] or [5] and references therein. We consider this model here in parallel to the second one which is less known in order to demonstrate the similarity between the two.

In general, the Casimir force can be represented as resulting from the ground state energy of the electromagnetic field

$$E_0 = \sum_{(n)} \frac{1}{2} \hbar \omega_{(n)}, \tag{2}$$

where the sum runs over all quantum numbers, i.e., over the whole physical spectrum. In the considered cases it consists of the photon modes which propagate in the whole space. They are labelled by a wave vector $\vec{k} \in \mathbb{R}^3$ and a discrete index for the two polarizations, TE and TM. In addition, for the TM polarization there are the surface plasmons which are characterized by a discrete imaginary component of the wave vector in the perpendicular direction, $k_z = i\kappa$.

In general, all degrees of freedom contribute to the sum (2). In the considered case these would be, in addition to the electromagnetic field, internal degrees of freedom of the media, say dipole oscillations of the molecules in the dielectric body in the first model and density oscillations of the charged fluid in the second model. These contributions are not considered in the present paper.

Of course, E_0 as given by equation (2) contains well-known divergences. These do not influence the Casimir force and can be removed for instance by subtracting the energy for infinite separation. After that (2) is finite and represented by a converging sum/integral over photons and plasmons:

$$E_0 \equiv E_{\text{total}} = E_{\text{plasmon}} + E_{\text{photon}}. \tag{3}$$

However, the integrand in E_{photon} is highly oscillating and very inconvenient to handle. Therefore, it is natural to move the integration path to the imaginary axis where the integrand becomes exponentially decreasing and where it is easy to perform the integration numerically. This is used in the Lifshitz formula too and gives E_{total} directly.

Below we calculate the vacuum energy E_{total} in this way for both models. For the first model we repeat known results, and for the second model the calculation is probably new. Then we calculate separately the plasmon contributions which is an easy exercise. From this the photon contribution is then $E_{\text{photon}} = E_{\text{total}} - E_{\text{plasmon}}$ and we can compare the relative weight of the individual contributions.

Technically our calculation is based on the formalism introduced in [6] (see also the review [5], section 5.1.1) where the vacuum energy is calculated in the background of a potential depending on one coordinate only. In this formulation, the handling of bound states is particularly transparent.

We start the next section with a description of the models and a collection of the necessary formulae. In section 3, we describe the formulae we use for the calculation of the vacuum energy. In section 4, the calculations are carried out and the results are presented. Section 5 deals with the special case of two attractive delta potentials and the final section contains the conclusions.

Starting from here we put $\hbar = 1$ and consider c as the speed of light relative to the vacuum.

2. Description of the models

Let us start from a scalar field Φ in a geometry of parallel planes. After Fourier transform in time and in the directions parallel to the planes we note the wave equation

$$\left(\frac{1}{c^2} \omega^2 - k_\parallel^2 + \frac{\partial^2}{\partial z^2} \right) \Phi(k_\parallel, z) = 0, \tag{4}$$

where c is the speed of light which may depend on the medium and $k_\parallel$ is the wave vector in parallel to the planes. This equation is to be supplemented by the corresponding matching conditions at $z = \pm \frac{L}{2}$. In the following, we need the scattering solutions to equation (4) which are those solutions which have for $z \to \pm\infty$ the asymptotic behaviour

$$\Phi(k_\parallel, z) \underset{z \to -\infty}{\sim} e^{ikz} + r \, e^{-ikz},$$
$$\underset{z \to \infty}{\sim} t \, e^{ikz}, \tag{5}$$

where r and t are the reflection and transmission coefficients and k is the wave number in the direction perpendicular to the planes. The frequency of these solutions follows from the dispersion relation $\omega^2/c^2 = k_\parallel^2 + k^2$.

A surface plasmon is a bound state in the one-dimensional problem and appears on the positive imaginary axis at $k = i\kappa$ with a wavefunction following from (5) after division by t:

$$\Phi_{\text{bs}}(k_\parallel, z) \underset{z \to -\infty}{\sim} \frac{r}{t} e^{\kappa z},$$
$$\underset{z \to \infty}{\sim} e^{-\kappa z}, \tag{6}$$

and t has a pole in $k = i\kappa$ whereas $\frac{r}{t}$ is finite in that point.

The matching conditions are as follows.

(i) *Dielectric bodies.* The field strength components $D_\perp$ and $E_\parallel$ must be continuous across the surface. The polarizations of the electromagnetic field can be separated into transverse electric (TE) and transverse magnetic (TM) ones. The matching conditions for the appearing scalar functions are

$$
\begin{aligned}
\Phi_+ - \Phi_- &= 0, & \Phi'_+ - \Phi'_- &= 0, &\text{(TE)} \\
\epsilon_+ \Phi_+ - \epsilon_- \Phi_- &= 0, & \Phi'_+ - \Phi'_- &= 0, &\text{(TM)}
\end{aligned}
\tag{7}
$$

where $\Phi_\pm$ and $\epsilon_\pm$ are the limiting values from the right and from the left. The dispersion with $\epsilon(\omega)$, equation (1), is

$$\omega = \sqrt{\omega_p^2 + k_\parallel^2 + k^2}.$$ (8)

The reflection and scattering coefficients are well known. We note them first for one surface assuming $\epsilon \neq 1$ on the left side and $\epsilon = 1$ on the right side. Because we have plane waves outside the surface we can write the wavefunction directly:

$$\Phi(k_\parallel, z) = \begin{cases} e^{ikz} + r\, e^{-ikz} & \text{for} \quad z < 0, \\ t\, e^{iqz} & \text{for} \quad z > 0, \end{cases}$$ (9)

where q is the wave vector in the perpendicular direction in between the planes and $\omega^2 = k_\parallel^2 + q^2$ holds. In these notations, the transmission and reflection coefficients are

$$t = \frac{2k}{k+q}, \qquad r = \frac{k-q}{k+q}, \qquad \text{(TE)}$$
$$t = \frac{2\epsilon k}{k+\epsilon q}, \qquad r = \frac{k-\epsilon q}{k+\epsilon q}. \qquad \text{(TM)}$$ (10)

(ii) *Plasma sheets.* For the plasma sheet the matching conditions demand $E_\parallel$ to be continuous across the surface and for the jump of the normal component discont $E_z = \frac{2\Omega}{\omega^2}\nabla_\parallel \cdot E_\parallel$ to hold. For the details of the model, we refer to [4]. The parameter Ω is proportional to the density of carriers in the plasma sheet and coincides with the parameter q in [4], equation (2.5).

In this model, the polarizations can be separated into TE and TM too and the matching conditions then read

$$\Phi_+ - \Phi_- = 0, \qquad \Phi'_+ - \Phi'_- = 2\Omega\Phi, \qquad \text{(TE)}$$
$$\Phi'_+ - \Phi'_- = 0, \qquad \Phi_+ - \Phi_- = -2\frac{\Omega}{\omega^2}\Phi'. \qquad \text{(TM)}$$ (11)

The matching condition for the TE polarization is just the same as for a delta function potential on a plane with strength Ω. The potential is repulsive because $\Omega > 0$ follows from the physics of the model. As known, a repulsive delta potential does not accommodate bound states and so there are no surface plasmons in this model. However, below we will investigate the unphysical case of $\Omega < 0$ too, where bound states are present. The matching condition for the TM polarization is sometimes associated with a δ' potential although this is not correct (see for example [7], appendix). The combination of parameters $-2\frac{\Omega}{\omega^2}$ appears from the physics of the model and leads to a bound state resp. to surface plasmons. In this model the dispersion is simply

$$\omega = \sqrt{k_\parallel^2 + k^2}$$ (12)

and the reflection and transmission coefficients are

$$t = \frac{ik}{ik - \Omega}, \qquad r = \frac{\Omega}{ik - \Omega}, \qquad \text{(TE)}$$
$$t = \frac{\omega^2}{\omega^2 + ik\Omega}, \qquad r = \frac{ik\Omega}{\omega^2 + ik\omega}. \qquad \text{(TM)}$$ (13)

For two parallel surfaces at distance L, the transmission coefficient written in terms of the coefficients from single surfaces is (a formula of this kind can be found e.g. in [8])

$$t = \frac{t_l t_r \exp(i(q-k)L)}{1 + \frac{t_l}{t_l}\bar{r}_l r_r\, e^{2iqL}},$$ (14)

where $\bar{t}$ is the complex conjugated to t and the indices indicate the left resp. right surface. This matters only for the first model where the coefficients (10) are written for the left surface. That for the right one follows by interchanging k and q and substituting $\epsilon \rightarrow 1/\epsilon$. In this way, for the dielectric model

$$
t = \frac{\frac{4kq}{(k+q)^2}}{1-\left(\frac{k-q}{k+q}\right)^2 \mathrm{e}^{2iqL}}, \quad \text{(TE)}
$$
$$
t = \frac{\frac{4\epsilon kq}{(k+\epsilon q)^2}}{1-\left(\frac{k-\epsilon q}{k+\epsilon q}\right)^2 \mathrm{e}^{2iqL}}, \quad \text{(TM)}
$$

(15)

holds and for the two plasma shells we have

$$
t = \frac{\frac{-k^2}{(ik-\Omega)^2}}{1-\left(\frac{\Omega}{ik-\Omega}\right)^2 \mathrm{e}^{2ikL}}, \quad \text{(TE)}
$$
$$
t = \frac{\frac{\omega^4}{(\omega^2+ik\Omega)^2}}{1+\left(\frac{\Omega k}{\omega^2+i\Omega k}\right)^2 \mathrm{e}^{2ikL}}. \quad \text{(TM)}
$$

(16)

The surface plasmons are given by the poles of t on the imaginary axis at $k = i\kappa$. For the dielectric model we get the equation

$$
\frac{\kappa + \epsilon q}{\kappa - \epsilon q} = -\sigma \, \mathrm{e}^{-qL}
$$

(17)

with $\epsilon = 1 - \frac{\omega_p^2}{\omega_p^2+k_\parallel^2-\kappa^2}$ and $q = \sqrt{k_\parallel^2 - \kappa^2}$. For $\sigma = +1$, the corresponding wavefunction is symmetric and for $\sigma = -1$ it is antisymmetric. The antisymmetric plasmon exists only for $k_\parallel \geqslant \omega_p/\sqrt{1 + \omega_p L/2}$. We denote the solutions of equation (17) by $\kappa_\sigma(k_\parallel, \omega_p, L)$. We introduce a similar notation for the frequency

$$
\omega(k_\parallel, k) = \sqrt{\omega_p^2 + k_\parallel^2 + k^2}.
$$

(18)

A single surface with ϵ given by (1) has one plasmon with

$$
\kappa_{\text{single}}(k_\parallel, \omega_p, L) = \sqrt{\frac{\omega_p^2}{2} + \sqrt{\left(\frac{\omega_p^2}{2}\right)^2 + k_\parallel^4}},
$$

(19)

which can be obtained from $\kappa_\sigma(k_\parallel, \omega_p, L)$ in the limit $L \rightarrow \infty$.

For the plasma shell model, we note the corresponding formulae. For the TE polarization, the equation for κ is

$$
1 + \frac{\kappa}{\Omega} = -\sigma \, \mathrm{e}^{-\kappa L}.
$$

(20)

It has solutions for $\Omega < 0$ only and the antisymmetric solution has the additional constraint $|\Omega| > 1/L$. For one plane the solution is $\kappa_{\text{single}} = -\Omega$. For the TM polarization the equation is

$$
\frac{\omega^2 - \kappa\Omega}{\kappa\Omega} = -\sigma \, \mathrm{e}^{-\kappa L}
$$

(21)

with $\omega = \sqrt{k_\parallel^2 - \kappa^2}$. The solutions exist for $\Omega > 0$. For one plane the solution is $\kappa_{\text{single}} = \frac{1}{2}\left(\sqrt{\Omega^2 + 4k_\parallel^2} - \Omega\right)$. Here we also introduce notations with an explicit indication of the arguments, $\kappa_\sigma(k_\parallel, \Omega, L)$ and

$$
\omega(k_\parallel, k) = \sqrt{k_\parallel^2 + k^2}.
$$

(22)

Finally we note that for the plasmons the inequalities

$$\kappa_- < \kappa_{\text{single}} < \kappa_+ < \omega_p \text{ (the dielectric plasma model)},$$

$$\kappa_- < \kappa_{\text{single}} < \kappa_+ < k_\parallel \text{ (the model of two plasma sheets)}$$

(23)

hold.

3. The vacuum energy

The vacuum energy for background potentials depending on one coordinate have been considered repeatedly. We follow here the derivation given in [6], or equivalently, in [5], section 5.1.1, but we change some notation, $s_{12} \to r$ and $s_{11} \to t$, for instance. The basic formula is

$$E_0 = \frac{1}{2} \int \frac{\mathrm{d}k_\parallel}{(2\pi)^2} \left\{ \sum_\sigma \omega(k_\parallel, \mathrm{i}\kappa_\sigma) + \int_0^\infty \frac{\mathrm{d}k}{2\pi\mathrm{i}} \omega(k_\parallel, k) \frac{\partial}{\partial k} \ln \frac{t(k)}{t(-k)} \right\}, \quad (24)$$

where $t(k)$ is the reflection coefficient defined in the scattering problem (5) and the sum runs over the surface plasmons. This formula is quite general and a dependence of ω and of $t(k)$ on $k_\parallel$ is allowed. By means of $\frac{t(k)}{t(-k)} = \mathrm{e}^{2\mathrm{i}\delta}$ it is equivalent to representations in terms of scattering phase shifts or the mode density.

In this formula, the ultraviolet divergences are still present. We remove them by subtracting the doubled contribution from one single surface. This is equivalent to subtracting the limit of large separation L. In the formulae for the transmission coefficients, equations (15) and (16), this results in removing the factors in the numerators and in the plasmon contributions we substitute

$$\omega(k_\parallel, \mathrm{i}\kappa_\sigma) \to \omega(k_\parallel, \mathrm{i}\kappa_\sigma) - \omega(k_\parallel, \mathrm{i}\kappa_{\text{single}}). \quad (25)$$

Obviously, we subtract distance independent terms which do not contribute to the Casimir force.

By means of equation (24) the vacuum energy is given as the sum (3) of plasmon and photon contributions and it consists of sum/integrals over the physical spectrum. While the (subtracted) plasmon contribution is fast converging, the (subtracted) photon contribution is highly oscillating. As already mentioned, it is sufficiently inconvenient to calculate it in this representation and we move the integration over k towards the imaginary axis. In that case a fast converging integral appears which can be evaluated numerically without much effort. This procedure is standard and well known. For instance, the integration over the imaginary axis is used in the Lifshitz formula although this is not always easy to see because of different choices of the variables in the integrations.

The only nontrivial moment which deserves attention is the role of the surface plasmons. In [6], it was shown that in the process of deformation the integration contour crosses poles of the logarithmic derivative in (24). These are just the poles which correspond to the bound states resp. to the surface plasmons. The crossings give extra contributions which just cancel the sum in (24). As a result one obtains the representation

$$E_0 \equiv E_{\text{total}} = \frac{1}{2} \int \frac{\mathrm{d}k_\parallel}{(2\pi)^2} \int_{k_0}^\infty \frac{\mathrm{d}k}{\pi} \omega(k_\parallel, \mathrm{i}k) \frac{\partial}{\partial k} \ln t(\mathrm{i}k), \quad (26)$$

for the *total* vacuum energy. The lower integration boundary for k is $k_0 = \sqrt{\omega_p^2 + k_\parallel^2}$ for the dielectric bodies with $\omega(k_\parallel, \mathrm{i}k)$ given by (18) and $k_0 = k_\parallel$ for the plasma sheets with $\omega(k_\parallel, \mathrm{i}k)$ given by (22). This formula is equivalent to the Lifshitz formula.

The surface plasmons exist in both models in the TM polarization only. However, in the TE polarization in the plasma shell model, which is equivalent to delta function potentials on the planes, the surface plasmons exist for a 'wrong' sign of the charge density parameter Ω, i.e., for $\Omega < 0$. Although in the considered model this case is not physical it is interesting to consider it anyway. We do this in section 5.

4. Calculation of the total vacuum energy and of the plasmon contributions

As shown in the preceding section the total vacuum energy is given by equation (26). For purposes of numerical evaluation it is meaningful to integrate by parts to get rid of the derivative. After that the formula for the total vacuum energy reads

$$E_{\text{total}} = -\frac{1}{2} \int \frac{\mathrm{d}k_{\parallel}}{(2\pi)^2} \int_{k_0}^{\infty} \frac{\mathrm{d}k}{\pi} \frac{k}{\omega(k_{\parallel}, ik)} \ln t(ik) \tag{27}$$

and this is the form which we use in the present section.

Now we collect the formulae for the transmission coefficients on the imaginary axis taking the subtractions into account. For the dielectric bodies

$$
\begin{aligned}
(t(ik))^{-1} &= 1 - \left(\frac{k-q}{k+q}\right)^2 \mathrm{e}^{-2qL}, &\quad \text{(TE)} \\
(t(ik))^{-1} &= 1 - \left(\frac{k-\epsilon q}{k+\epsilon q}\right)^2 \mathrm{e}^{-2qL}, &\quad \text{(TM)}
\end{aligned}
\tag{28}
$$

holds where q and ϵ must be expressed in terms of $k_{\parallel}$ and k, i.e., as $q = \sqrt{k^2 - \omega_p^2}$ and $\epsilon = 1 - \frac{\omega_p^2}{\omega_p^2 + k_{\parallel}^2 - k^2}$. For the plasma sheets we have

$$
\begin{aligned}
(t(ik))^{-1} &= 1 - \left(\frac{\Omega}{k+\Omega}\right)^2 \mathrm{e}^{-2kL}, &\quad \text{(TE)} \\
(t(ik))^{-1} &= 1 - \left(\frac{\Omega k}{k_{\parallel}^2 - k^2 - \Omega k}\right)^2 \mathrm{e}^{-2kL}. &\quad \text{(TM)}
\end{aligned}
\tag{29}
$$

Inserting these formulae into (27) the calculation of the vacuum energy is an easy numerical exercise. For $L \to \infty$ in all cases the ideal conductor limes $E = -\pi^2/240 \, L^{-3}$ is reobtained.

Note that equation (27) contains a double integral. In all cases one integration can be carried out explicitly. In both models for the TE polarizations this is trivial because $t(ik)$ does not depend on $k_{\parallel}$. For the TM polarizations after changing the variables from $k_{\parallel}$ to ω the integration over ω can be carried out explicitly. This integration can be done by standard computer algebra programs and the result is a quite lengthy expression which we do not display here. Within the dielectric model the corresponding integration was done for the force, i.e., for the derivative of equation (27) with respect to the distance, in the appendix in [9].
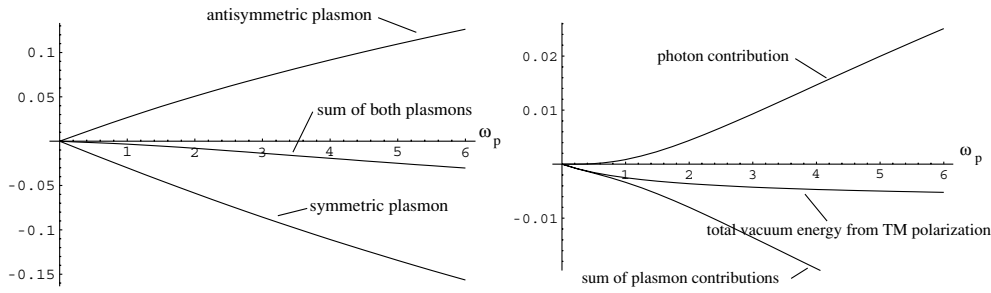
The contribution of the surface plasmons follows from equations (24) and (25). It reads

$$E_{\text{plasmon}} = \frac{1}{2} \int \frac{\mathrm{d}k_{\parallel}}{(2\pi)^2} \sum_{\sigma=\pm 1} \left\{ \sqrt{\omega_p^2 + k_{\parallel}^2 - \kappa_{\sigma}^2(k_{\parallel}, \omega_p, L)} - \sqrt{\omega_p^2 + k_{\parallel}^2 - \kappa_{\text{single}}^2(k_{\parallel}, \omega_p, L)} \right\} \tag{30}$$
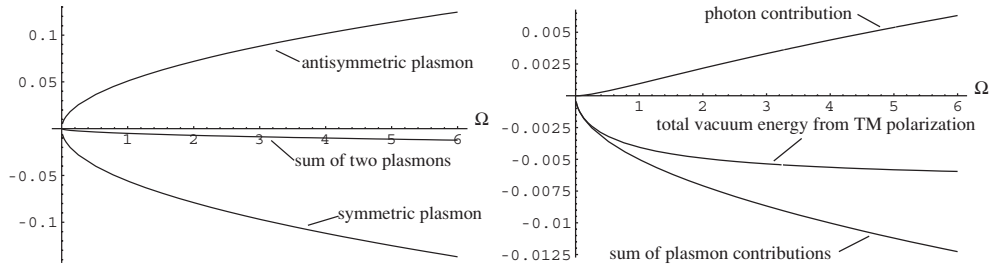
for the dielectric bodies with the constraint $k_{\parallel} \geqslant \omega_p/\sqrt{1 + \omega_p L/2}$ for $\sigma = -1$ and

$$E_{\text{plasmon}} = \frac{1}{2} \int \frac{\mathrm{d}k_{\parallel}}{(2\pi)^2} \sum_{\sigma=\pm 1} \left\{ \sqrt{k_{\parallel}^2 - \kappa_{\sigma}^2(k_{\parallel}, \Omega, L)} - \sqrt{k_{\parallel}^2 - \kappa_{\text{single}}^2(k_{\parallel}, \Omega, L)} \right\} \tag{31}$$

Figure 1. The vacuum energy of the TM polarization in the dielectric plasma model and its several constituents, observe the large compensation in both panels.

Figure 2. The vacuum energy of the TM polarization in the plasma sheet model and its several constituents, observe the large compensation in both panels.

for the plasma sheets. All these quantities are shown in figure 1 for the dielectric bodies as a function of the plasma frequency ω_p and in figure 2 for the plasma sheets as a function of the parameter Ω for a fixed distance $L = 1$. In both cases in the left panel we observe a significant compensation between the two plasmon contributions. In the right panel we see a compensation between the plasmon and the photon contributions.

The vacuum energy can be represented in the form

$$E_{\text{total}} = -\frac{\pi^2}{240} \frac{1}{L^3} f(\omega_p L), \tag{32}$$

where the factor in front is the ideal conductor contribution and the dimensionless function $f(\omega_p L)$ describes the deviation from the ideal conductor in the considered model. In the case of the plasma sheet model it depends on ΩL. For large argument it has the limit

$$\lim_{x \to \infty} f(x) = -1. \tag{33}$$

For both models and both polarizations this function is shown in figure 3.

Taking into account the dependence of the vacuum energy on the parameters as given by equation (32) we see from figures 1 and 2 that the relative compensation between the plasmons and the photon contribution increases with distance L. This can be discussed in more detail. We consider the individual plasmon contributions as given by equations (30) and (31) and represent these formulae in the form

$$E_{\text{plasmon},\sigma} = \frac{1}{4\pi} \int_{k_\parallel^0}^{\infty} dk_\parallel k_\parallel \omega_\sigma(\omega_p, L, k_\parallel) \tag{34}$$
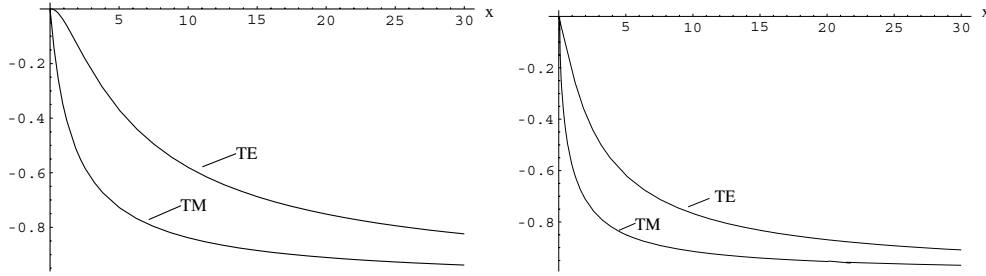
Figure 3. The contributions from the TE and TM polarizations to the vacuum energy in the dielectric model (left panel) and in the plasma sheet model (right panel) normalized to the perfect conductor limes.
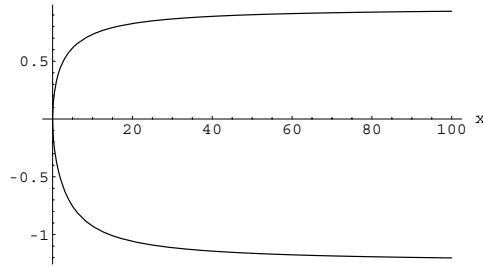
Figure 4. The function $g(x)$ as defined in equation (38) in the dielectric plasma model, the lower curve is for the symmetric plasmon and the upper curve is for the antisymmetric plasmon.

with $k_\parallel^0 = \omega_p/\sqrt{1 + \omega_p L/2}$ for $\sigma = -1$ in the dielectric model and $k_\parallel^0 = 0$ otherwise. In (34) we introduced the notation

$$\omega_\sigma(\omega_p, L, k_\parallel) = \sqrt{\omega_p^2 + k_\parallel^2 - \kappa_\sigma(k_\parallel, \omega_p, L)} - \sqrt{\omega_p^2 + k_\parallel^2 - \kappa_{\text{single}}(k_\parallel, \omega_p, L)}. \tag{35}$$

$\kappa_\sigma(k_\parallel, \omega_p, L)$ are given by equations (17) and (21) and $\kappa_{\text{single}}(k_\parallel, \omega_p, L)$ by equation (19). Now we make the substitution $k_\parallel \to k_\parallel/L$ in the integral in (34):

$$E_{\text{plasmon},\sigma} = \frac{1}{4\pi L^3} \int_{k_\parallel^0 L}^\infty \mathrm{d}k_\parallel k_\parallel L \omega_\sigma(\omega_p, L, k_\parallel/L). \tag{36}$$

For dimensional reasons the expression $L\omega_\sigma(\omega_p, L, k_\parallel/L)$ depends on the product $\omega_p L$ only. This is a representation in parallel to (32) and the integral in (36) is a dimensionless function on $\omega_p L$. In the next step we substitute $k_\parallel \to k_\parallel \sqrt{\omega_p L}$ and represent the plasmon contribution in the form

$$E_{\text{plasmon},\sigma} = \frac{\sqrt{\omega_p L}}{4\pi L^3} g(\omega_p L) \tag{37}$$

with

$$g(\omega_p L) = \sqrt{\omega_p L} \int_{k_\parallel^0 \sqrt{L/\omega_p}}^\infty \mathrm{d}k_\parallel k_\parallel L \omega_\sigma(\omega_p, L, k_\parallel \sqrt{\omega_p/L}). \tag{38}$$

So far this is a simple rewriting of the plasmon contribution. Now the statement is that

- for the dielectric model the function $g(x)$ has a finite limit for $x \to \infty$, namely

$$\lim_{x\to\infty} g(x) = \begin{cases} -1.2448 & \text{for} \quad \sigma = 1 \quad \text{(symmetric plasmon)}, \\ 0.9652 & \text{for} \quad \sigma = -1 \quad \text{(antisymmetric plasmon)}. \end{cases} \tag{39}$$

The function $g(x)$ is shown in figure 4 for both plasmons.

- for the plasma sheets the function $g(x)$ does not depend on its argument at all. The values are

$$g(x) = \begin{cases} -0.702\,427 & \text{for} \quad \sigma = 1 \quad \text{(symmetric plasmon),} \\ 0.639\,449 & \text{for} \quad \sigma = -1 \quad \text{(antisymmetric plasmon).} \end{cases} \tag{40}$$

This is the behaviour of the plasmon contributions for large both or one of the two, distance L and plasma frequency ω_p (in the second model ω_p must be substituted by Ω). We see that the plasmon contributions grow by a factor $\sqrt{\omega_p L}$ faster than the total vacuum energy. At once, by means of equation (3), this gives us the asymptotic behaviour of the photon contribution and we see how the mentioned compensation is growing.

It should be mentioned that in this way we got at once in the plasma sheet model the behaviour of the plasmon contributions for small Ω and for small L.

We note that the statements of equations (39) and (40) are obtained numerically. No attempt had been made to get these results analytically. For the dielectric plasma model similar results are reported in [2].

In a similar manner the asymptotic behaviour for small separations can be obtained numerically. In the dielectric model we find

$$E_{\text{TE}} \sim -0.002\omega_p^3, \qquad\qquad E_{\text{TM}} \sim -0.0039\frac{\omega_p}{L^2},$$
$$E_{\text{plasmon},\sigma=+1} \sim -0.0305\frac{\omega_p}{L^2}, \qquad E_{\text{plasmon},\sigma=-1} \sim 0.0267\frac{\omega_p}{L^2}. \tag{41}$$

This confirms the known behaviour that in this limit the TE contribution is small and the TM contribution is the sum of the two plasmon contributions.

In the plasma sheet model we find

$$E_{\text{TE}} \sim -0.013\frac{\Omega^2}{L}, \qquad E_{\text{TM}} \sim -0.005\,01\frac{\sqrt{\Omega}}{L^{5/2}}.$$

For the plasmons, by means of equation (40), we know the behaviour for all distances

$$E_{\text{plasmon},\sigma=+1} = 0.050\,90\frac{\sqrt{\Omega}}{L^{5/2}},$$
$$E_{\text{plasmon},\sigma=-1} = -0.055\,89\frac{\sqrt{\Omega}}{L^{5/2}}. \tag{42}$$

So in this model we observe the same pattern; the TE contribution is small and the TM contribution is at small separation the sum of the two plasmon contributions.

5. Two attractive delta potentials

As mentioned above for the wrong sign of Ω in the plasma sheet model, the TE polarization has the surface plasmons too. This is equivalent to a scalar problem with attractive delta potentials on two planes. A single attractive delta potential has always one bound state. For two planes there are two of them, provided $L > \Omega$ holds otherwise there is only one. Further we note that, because the considered quantum field is massless, any bound state makes the potential instable, i.e., it causes particle creation. As a consequence, the vacuum energy acquires an imaginary part. In general, there is nothing special about that and we include this case here only because we are discussing the role of the surface plasmons.

There are two ways to perform the calculations. The first is to make an analytic continuation in equation (26). When moving the binding energy κ_σ of a plasmon which by means of (23) is initially below the integration part upwards so that κ_σ becomes larger than k_0 one needs to deform the integration path in order for it not to hit the pole. Then the integral remains finite but it acquires an imaginary part.

The second way is to calculate the plasmon and photon contributions separately. Then it is obvious that the photon contribution is real and the plasmon contribution delivers just the imaginary part. Indeed, consider a plasmon contribution

$$E_{\text{plasmon}} = \frac{1}{2} \int \frac{dk_\parallel}{(2\pi)^2} \sqrt{k_\parallel^2 - \kappa^2}, \tag{43}$$

where κ is a solution of equation (20). As usual, we are concerned with a divergent quantity. Here the subtraction (25) of the contributions of single planes does not work because κ does not depend on $k_\parallel$. So we take a standard regularization, say

$$E_{\text{plasmon}}^\delta = \frac{1}{2} \int \frac{dk_\parallel}{(2\pi)^2} \sqrt{k_\parallel^2 - \kappa^2} e^{-\delta\sqrt{k_\parallel^2 - \kappa^2}}. \tag{44}$$

This integral can easily be calculated explicitly

$$E_{\text{plasmon}}^\delta = \frac{1}{2\pi\delta^3} + \frac{i\kappa^3}{12\pi} + O(\delta). \tag{45}$$

After dropping the distance independent divergent part we just get

$$E_{\text{plasmon}} = \frac{i\kappa^3}{12\pi}, \tag{46}$$

which is the contribution of one surface plasmon for two attractive delta potentials.

In the photon contribution we proceed in the same way as before. We move the integration path for k and contract it on the imaginary axis. Now we have for $k_\parallel < \kappa$ a pole on the cut. It can be shown that after the contraction of the path, for example using the well-known formula

$$\frac{1}{x - i0} = \text{Vp}\frac{1}{x} + i\pi\delta(x),$$

the imaginary part disappears and only the value principal integral remains which is real. Actually this happens because the factor $\left(k^2 + k_\parallel^2\right)^{\frac{1}{2}}$ has different signs on both sides of the cut. After that the order of integrations can be interchanged and the integration over $k_\parallel$ can be carried out. Finally, integrating by parts, the photon contribution can be written in the form

$$E_{\text{photon}} = \frac{-\pi^2}{1440L^3} h(2\Omega L) \tag{47}$$

with

$$h(x) = \frac{45}{\pi^4}\text{Vp}\int_0^\infty dk\, k^2 \log\left|\frac{1}{1 - \left(\frac{x}{k+x}\right)^2 e^{-k}}\right|. \tag{48}$$

The function $h(x)$ is shown in figure 5. It has the property $\lim_{x \to \pm\infty} h(x) = 1$. For $x > 0$ it represents the deviation of the vacuum energy of the TE polarization in the plasma sheet model from the ideal conductor limit and for $x < 0$ it represents the deviation of the real part of the vacuum energy of two attractive delta potentials. We observe two regions with an attractive force and one with a repulsive force.
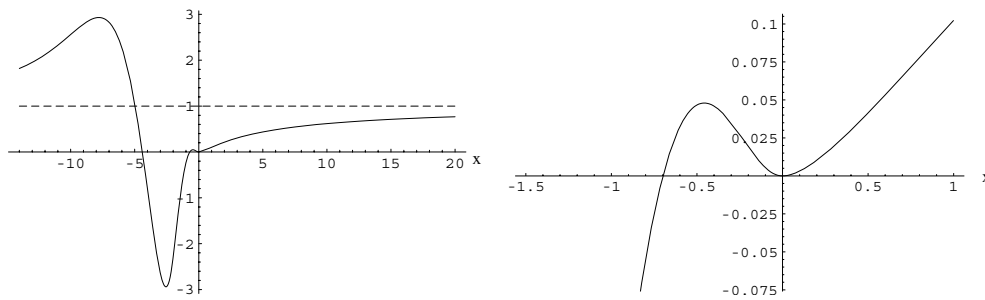
Figure 5. In the left panel the function $h(x)$, (48), which describes the vacuum energy relative to the ideal conductor limes for two delta potentials which are attractive for negative x and repulsive for positive x where they coincide with the TE polarization in the plasma sheet model, in the right panel the same function for small x.

6. Conclusions

We calculated the vacuum energy in two models involving surface plasmons. As a result we are able to support the physically instructive picture that at small distances the plasmon contributions dominate the vacuum energy. However, there is a compensation by one order of magnitude between the two plasmon contributions (see equations (41) and (42)). At large separation we find that the compensation between the plasmon contributions grows and that in addition there is a large and growing compensation between the sum of the two plasmon contributions and the photon contribution.

In the dielectric model the compensations were found in the paper [2] where the importance of the plasmon contributions was emphasized. We would like to mention that the vacuum energy which is a relatively small quantity as compared to the plasmon contributions (even for small separations there is a compensation between the individual plasmons, see equation (41)) can be directly calculated in the known representation where the momentum integration in the direction perpendicular to the surfaces goes over the imaginary axis. In that formula large compensations are absent. A possible disadvantage, however, is that one needs to know $\epsilon(\omega)$ for imaginary ω.

Also we considered the example of two planes carrying attractive delta potentials. Here the surface plasmons do not contribute to the real part of the vacuum energy. They deliver the imaginary part which is present because in that model particle creation is present. It is obvious that the imaginary part is present for large separations too.

In general, the representations of the vacuum energy by an integral over the imaginary axis on the one hand and as a sum of plasmon and photon contributions are connected by the dispersion relation the transmission coefficient $t(k)$ obeys, see for example equation (27) in [6]. To some extent this is equivalent to the Kramers–Kronig relation for $\epsilon(\omega)$. Both are exact relations and therefore a large compensation does not present a problem. However, if trying to make some kind of approximation in these relations the compensation may become a source of large errors.

Acknowledgments

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